ELSEVIER Journal of Geometry and Physics 19 (1996) 90–98

JOURNAL OF GEOMETRY AND PHYSICS

Reduction of Poisson–Nijenhuis manifolds

Izu Vaisman [1]

Department of Mathematics and Computer Science, University of Haifa, Israel

Received 18 October 1994

Abstract

The Poisson reduction theorem of Marsden–Raţiu is refined so as to include the reduction of either a Poisson–Nijenhuis structure (Magri and Morosi, 1984) or a complementary 2-form (Vaisman, 1995).

Keywords: Poisson–Nijenhuis structures; Complementary 2-forms; Reduction
1991 MSC: 58F05, 53C15

1. Introduction

Our general framework is the C^∞-category, and the basic objects studied in this note are *Poisson manifolds* and their associated *Nijenhuis structures* and *complementary 2-forms.*

A *Poisson manifold* is a differentiable manifold M^m endowed with a *Poisson bivector,* the latter being a bivector field P on M which satisfies $[P, P] = 0$. The bracket is the so-called *Schouten–Nijenhuis bracket,* and the existence of P is equivalent to the existence of the *Poisson bracket* $\{f, g\}$ $(f, g \in C^\infty(M))$ which makes $C^\infty(M)$ into a Lie algebra, and which is a derivation of $C^\infty(M)$ if either f or g is fixed. We assume that the reader is familiar with the theory of Poisson manifolds (e.g., [7]).

A *Nijenhuis tensor* A is a $(1, 1)$-tensor field of M which has a vanishing *Nijenhuis torsion*:

$$\mathcal{N}_A(X,Y) := [AX, AY] - A[AX, Y] - A[X, AY] + A^2[X, Y] = 0 \tag{1.1}$$

$(X, Y$ are vector fields on $M)$. If P is a Poisson bivector and A is a Nijenhuis tensor, the pair (P, A) is called a *Poisson–Nijenhuis structure,* and (M, P, A) is a *Poisson–Nijenhuis manifold,* provided that the following two conditions hold:

$$P(\alpha \circ A, \beta) = P(\alpha, \beta \circ A) \tag{1.2}$$

[1] E-mail: i.vaisman@uvm.haifa.ac.il.

SSDI 0393-0440(95)00024-0

for any two 1-forms α, β of M, and

$$C_{(P,A)}(\alpha, X, \beta) := \beta((L_{\sharp_P \alpha} A)X) - \alpha((L_{\sharp_P \beta} A)X) + (AX)(P(\alpha, \beta)) - X(P(\alpha \circ A, \beta)) = 0 \tag{1.3}$$

$\forall X$, α, β as in (1.1) and (1.2). $C_{(P,A)}$ is called the *Schouten invariant*, L denotes Lie derivatives, and $\sharp_P \alpha$ is defined by $\beta(\sharp_P \alpha) = P(\alpha, \beta)$. The importance of the Poisson–Nijenhuis structures comes from their role in the study of the integrability of Hamiltonian dynamical systems (see [2,5], etc.).

Furthermore, if (M, P) is a Poisson manifold, the bracket $\{\mathrm{d}f, \mathrm{d}g\} := \mathrm{d}\{f, g\}$ extends to a Lie bracket of 1-forms which has the general expression (e.g., [3,7]):

$$\{\alpha, \beta\}_P = \mathrm{d}(P(\alpha, \beta)) - i(\sharp_P \alpha)\, \mathrm{d}\beta - i(\sharp_P \beta)\, \mathrm{d}\alpha. \tag{1.4}$$

Moreover, by the same algebraic machinery as for the usual Schouten–Nijenhuis bracket, the bracket (1.4) extends to arbitrary differential forms [4,3,7]. A 2-form ω on M is a *complementary 2-form* of P, and (M, P, ω) is a *complemented Poisson manifold* [9] if ω satisfies

$$\{\omega, \omega\}_P = 0. \tag{1.5}$$

We have proven in [9] that if ω is a closed complementary 2-form of P, and if we define as usual $\flat_\omega(X) := i(X)\omega$, then $(P, A := \sharp_P \circ \flat_\omega)$ is a Poisson–Nijenhuis structure on M.

Now, we assume that the reader is familiar with the *symplectic reduction procedure* of Marsden and Weinstein (e.g., [1]). This procedure was extended by Marsden and Raţiu [6,10,7] to Poisson manifolds. Even prior to this, particular cases of reduction were used to reduce Poisson–Nijenhuis structures to the "kernel-free" situations needed in integrability theory [5].

The aim of the present note is to use the Marsden–Raţiu reduction theorem in order to deduce general reduction theorems for Poisson–Nijenhuis manifolds and for complemented Poisson manifolds. In particular, we get reductions via Hamiltonian group actions and momentum maps. For instance, if the connected Lie group G has a Hamiltonian action on the Poisson manifold (M, P), with equivariant momentum map $J : M \to \mathcal{G}^*$ ($\mathcal{G}$ is the Lie algebra of G and $\mathcal{G}^*$ is its dual), any solution of the Yang–Baxter equation of $\mathcal{G}$ lifts to a closed complementary 2-form ω of (M, P) which has reductions to the level sets of the regular values of J. Of course, the same holds for the Poisson–Nijenhuis structure defined by P and ω on M.

2. Reduction of the Poisson–Nijenhuis structures

Let (M, P) be a Poisson manifold, $\iota_N : N \subseteq M$ a submanifold of M, and E a vector subbundle of $TM|_N$ such that: (i) $E \cap TN = T\mathcal{F}$, where $\mathcal{F}$ is the foliation of N by the fibers of a submersion $\pi : N \to Q$; (ii) $\forall\varphi, \psi \in C^{\infty}(M)$ such that $\mathrm{d}\varphi|_E = \mathrm{d}\psi|_E = 0$ one has $\mathrm{d}\{\varphi, \psi\}|_E = 0$; (iii) there exists a Poisson structure P' on Q such that

$$\{\varphi, \psi\} \circ \iota_N = \{f, g\}' \circ \pi \tag{2.1}$$

$\forall f, g \in C^{\infty}(Q)$, and where φ, ψ are extensions of $f \circ \pi, g \circ \pi$ to M such that $\mathrm{d}\varphi|_E = \mathrm{d}\psi|_E = 0$. Then the Poisson manifold (Q, P') is said to be obtained by the *reduction of* (M, P) via (N, E).

The reduction theorem of Marsden–Raţiu [6] says that if conditions (i) and (ii) are satisfied, the reduced structure P' exists (i.e., (iii) holds) iff

$$\sharp_P Ann\, E \subseteq TN + E, \tag{2.2}$$

where $Ann\, E$ is the *annihilator* of E (i.e., the subbundle of $T^*M|_N$ which vanishes on E).

Now, using this result, we prove:

Theorem 2.1. *Let (M, P, A) be a Poisson–Nijenhuis manifold, N a submanifold, and E a vector subbundle of $TM|_N$ such that conditions* (i) *and* (ii) *described above hold. Moreover, assume that $A(TN) \subseteq TN$, $A(E) \subseteq E$, and that $A|_N$ sends $\mathcal{F}$-foliated vector fields to $\mathcal{F}$-foliated vector fields. Then, A projects by π to a $(1,1)$-tensor field A' of Q and, if*

$$\sharp_P Ann\, E \subseteq TN \tag{2.3}$$

also holds, (Q, P', A'), with P' of (2.1), *is again a Poisson–Nijenhuis manifold, and it is said to have been obtained from (M, P, A) by reduction via (N, E).*

Proof. On a foliated manifold, a foliated vector field is a vector field which preserves the foliation. Then, the hypotheses on A clearly imply the existence of A' and the fact that A' is a Nijenhuis tensor on Q.

Now, (2.3) implies (2.2), and P' also exists. In view of (2.1), the bivector P' of Q is determined by

$$P'_q(\lambda, \mu) = P_n(\widetilde{\pi^*\lambda}, \widetilde{\pi^*\mu}), \tag{2.4}$$

where $q \in Q$, $n \in N$, $\pi(n) = q$, $\lambda, \mu \in T^*Q$, and $\widetilde{\pi^*\lambda}, \widetilde{\pi^*\mu}$ are extensions of $\pi^*\lambda$, $\pi^*\mu$ from TN to $TM|_N$ such that $\widetilde{\pi^*\lambda}|_E = \widetilde{\pi^*\mu}|_E = 0$. Indeed, (2.1) and (2.4) are the same if $\lambda = \mathrm{d}f$, $\mu = \mathrm{d}g$, $f, g \in C^{\infty}(Q)$. Then, if we use the extensions $\pi^*\widetilde{(\lambda \circ A')} = \widetilde{\pi^*\lambda} \circ A$, $\pi^*\widetilde{(\mu \circ A')} = \widetilde{\pi^*\mu} \circ A$, we see that (1.2) implies $P'(\lambda \circ A', \mu) = P'(\lambda, \mu \circ A')$.

Therefore, we only have to check that the Schouten invariant $C_{(P', A')}$ defined by (1.3) vanishes. But, as shown in [8], we may see that condition $C_{(P', A')} = 0$ is equivalent to

$$\mathrm{d}g \circ L_{X'_f} A' = i(X'_g)\, \mathrm{d}(\mathrm{d}f \circ A') \tag{2.5}$$

$\forall f, g \in C^{\infty}(Q)$, and for their P'-Hamiltonian vector fields X'_f, X'_g. Moreover, it is enough to check that the lift of (2.5) by π^* holds.

Let φ, ψ be extensions of $f \circ \pi, g \circ \pi$ to M such that $d\varphi|_E = d\psi|_E = 0$. Since $C_{(P,A)} = 0$, we have the corresponding equality (2.5), i.e.,

$$d\psi \circ L_{X_\varphi} A = i(X_\psi)\, d(d\varphi \circ A). \tag{2.6}$$

We also have $X_\varphi(n) = \sharp_P\, d_n\varphi \in \sharp_P Ann\ E \subseteq TN$ $(n \in N)$ (because of (2.3)) and, as a consequence of this fact, (2.1) implies

$$X'_f(\pi(n)) = \pi_* X_\varphi(n) \quad (n \in N). \tag{2.7}$$

Now, let us take $Z_n \in T_n N$, and let Z be an $\mathcal{F}$-foliated vector field of N such that $Z(n) = Z_n$. It follows that

$$\begin{aligned} \pi^*(dg \circ L_{X'_f} A')(Z_n) &= d_{\pi(n)} g([X'_f, A'\pi_* Z] - A'[X'_f, \pi_* Z]) \\ &\overset{(2.7)}{=} d_{\pi(n)} g(\pi_*[X_\varphi|_N, AZ] - \pi_* A[X_\varphi|_N, Z]) \\ &= (d\psi \circ L_{X_\varphi} A)(Z_n) \\ &\overset{(2.6)}{=} d(d\varphi \circ A)(X_\psi(n), Z_n) \\ &= \pi^*(d(df \circ A'))(X_\psi(n), Z_n) \\ &= d(df \circ A')(X'_g(\pi(n)), \pi_* Z_n) \\ &= [i(X'_g(\pi(n)))\, d(df \circ A')](\pi_* Z_n) \\ &= \pi^*[i(X'_g\, d(df \circ A')](Z_n). \end{aligned}$$

(In this computation one should always keep in mind that the vector fields X_φ, X_ψ are tangent to N, a fact that was ensured by hypothesis (2.3).) □

Remark 2.2. In Theorem 2.1, the condition that $A|_N$ sends foliated vector fields to foliated vector fields may be replaced by the equivalent condition

$$[L_V(A|_N)](Y) \in T\mathcal{F}, \quad \forall V \in T\mathcal{F}, \ \forall Y \in TN. \tag{2.8}$$

Indeed, since it is enough to look at (2.8) for $Y_{n_0} \in T_{n_0}N$ $(n_0 \in N)$, we may assume that the vector field Y which extends Y_{n_0} to N is an $\mathcal{F}$-foliated field. Then

$$[L_V(A|_N)](Y) = [V, AY] - A[V, Y]$$

and $[V, Y] \in T\mathcal{F}$. Hence, $L_V(A|_N)(Y)$ and $[V, AY]$ either belong or do not belong to $T\mathcal{F}$, simultaneously.

For instance, (2.8) holds if we ask that

$$(L_X A)(TN) \subseteq E \tag{2.9}$$

for any vector field X of M such that $X|_N \in E$.

An interesting case of Poisson reduction is obtained if $E = \sharp_P Ann\ TN$, and the following hypotheses hold: (1) $rank\ P|_N = const.$, and N is transversal to the symplectic leaves of P; (2) $dim((\sharp_P Ann\ TN) \cap TN) = const.$ Notice that (2.3) is true in this case. Now, if we also have the Nijenhuis tensor A such that (P, A) is a Poisson–Nijenhuis structure, and if we add two more conditions: (3) $A(TN) \subseteq TN$; (4) the relation (2.9) is satisfied, we get a case of Poisson–Nijenhuis reduction. Indeed, (1.2) implies that $A\sharp_P\alpha = \sharp_P(\alpha \circ A)$ for any 1-form α, and then we see that condition (3) also implies $A(E) \subseteq E$. Thus, we may apply Theorem 2.1, and we get the desired result.

Finally, we note that all the cases of the Poisson–Nijenhuis reductions used in [5] are particular cases of Theorem 2.1.

3. Reduction of complementary 2-forms

Let us consider again a Poisson manifold (M, P), a submanifold N of M, and a vector subbundle $E \subseteq TM|_N$ such that conditions (i)–(iii) of the beginning of Section 2 are satisfied, and the reduced Poisson manifold (Q, P') of (M, P) via (N, E) exists. Furthermore, let ω be a complementary 2-form of P on M. Then, the following theorem provides conditions for the reducibility of ω to a complementary 2-form ω' of (Q, P').

Theorem 3.1. *Let $(M, P, N, E, \omega, Q, P')$ be as above, and let $\mathcal{F}$ be the foliation $E \cap TN$ of N. Assume that:* (i) *$\sharp_P \mathrm{Ann}\, E \subseteq TN$; $\sharp_P \mathrm{Ann}\, TN \subseteq E$; $\flat_\omega(TN) \subseteq \mathrm{Ann}\, E$;* (ii) *$\iota_N^*\omega$ is an $\mathcal{F}$-foliated (i.e., projectable) 2-form of N ($\iota_N^* : N \subseteq M$, and $\flat_\omega : TM \to T^*M$ is given by $\flat_\omega X := i(X)\omega$, $X \in TM$). Then, the projected 2-form ω' of ω onto Q is a complementary 2-form of (Q, P').*

Proof. We begin by proving an auxiliary result, namely, for the reduced Poisson manifold (Q, P'), one has

$$\iota_N^*\{\widetilde{\pi^*\lambda}, \widetilde{\pi^*\mu}\}_P = \pi^*\{\lambda, \mu\}_{P'}, \tag{3.1}$$

where the notation is that of formula (2.4), and the brackets of 1-forms are those defined by (1.4). Indeed, it is easy to understand that (2.4) implies

$$\iota_N^*(\mathrm{d}(P(\widetilde{\pi^*\lambda}, \widetilde{\pi^*\mu}))) = \pi^*(\mathrm{d}(P'(\lambda, \mu))). \tag{3.2}$$

Hence, in view of (1.4), (3.1) will be proven if we prove

$$\pi^*(i(\sharp_{P'}\lambda)\, \mathrm{d}\mu) = \iota_N^*(i(\sharp_P\widetilde{\pi^*\lambda})\, \mathrm{d}(\widetilde{\pi^*\mu})). \tag{3.3}$$

(The final terms of the brackets (3.1) are of the same form, but with the exchanged roles of λ and μ.) Since $\sharp_P(\widetilde{\pi^*\lambda}) \in TN$, because of the first part of hypothesis (i), the definition of the differential π_* of π yields

$$\pi_*(\sharp_P\widetilde{\pi^*\lambda}) = \sharp_{P'}\lambda. \tag{3.4}$$

Then, for any $\mathcal{F}$-foliated vector field Y of N, we get

$$\begin{aligned} \pi^*(i(\sharp_{P'}\lambda)\, \mathrm{d}\mu)(Y) &= \mathrm{d}\mu(\sharp_{P'}\lambda, \pi_*Y) = \mathrm{d}\mu(\pi_*\sharp_P(\widetilde{\pi^*\lambda}), \pi_*Y) \\ &= (\pi^*\, \mathrm{d}\mu)(\sharp_P\widetilde{\pi^*\lambda}, Y) = \mathrm{d}(\widetilde{\pi^*\mu})(\sharp_P\widetilde{\pi^*\lambda}, Y), \end{aligned} \tag{3.5}$$

where the last equality holds since (in view of the first part of hypothesis (i) only values of $\widetilde{\pi^*\mu}$ on tangent vectors of N appear in the evaluation of the last term of (3.5). Now, it is clear that (3.5) implies (3.3), because the latter is to be checked only for $Y_n \in T_nN$ $(n \in N)$, and all such Y_n may be extended to an $\mathcal{F}$-foliated field Y of N.

Now, by a general algebraic Schouten–Nijenhuis bracket formula [2,3], one has

$$\{\omega,\omega\}_P(X_m, Y_m, Z_m) = \sum_{Cycl(X,Y,Z)} \langle \flat_\omega L^*_{\flat_\omega X} Y, Z\rangle, \tag{3.6}$$

where $X_m, Y_m, Z_m \in T_mM$ $(m \in M)$, X, Y, Z are vector fields of M which extend X_m, Y_m, Z_m, and L^* is the Lie derivative operation of the Lie algebroid T^*M of bracket (1.4), and with the anchor map $\sharp_P$. From (3.6) we get

$$\begin{aligned}\{\omega,\omega\}_P(X_m, Y_m, Z_m) &= -\sum_{Cycl(X,Y,Z)} \langle L^*_{\flat_\omega X} Y, i(Z)\omega\rangle \\ &= \sum_{Cycl(X,Y,Z)} (\sharp_P \flat_\omega X_m)(\omega(Y,Z)) + \langle Y_m, \{\flat_\omega X, \flat_\omega Z\}_P\rangle.\end{aligned} \tag{3.7}$$

Let us take $m \in N$, $X_m, Y_m, Z_m \in T_mN$, and X, Y, Z extension vector fields which are tangent to N, and are foliated with respect to $\mathcal{F}$ on N. Then, the left-hand side of (3.7) is an evaluation of $\iota_N^*(\{\omega,\omega\}_P)$.

On the other hand, because of the last part of hypothesis (i) and of hypothesis (ii), we have

$$\langle Y_m, \{\flat_\omega X, \flat_\omega Z\}_P\rangle = \langle Y_m, \iota_N^*\{\flat_\omega X, \flat_\omega Z\}_P\rangle \overset{(3.1)}{=} \langle Y_m, \pi^*\{\flat_{\omega'}\pi_* X, \flat_{\omega'}\pi_* Z\}\rangle.$$

Furthermore, let us denote $\sharp_P \circ \flat_\omega = A$, $\sharp_{P'} \circ \flat_{\omega'} = A'$. Then, the first and last conditions of (i) yield $AX \in TN$, and this shows that the first term on the right-hand side of (3.7) is also an evaluation on N.

Moreover, let $\pi^*\widetilde{\flat_{\omega'}}\pi_*X$ be an extension of $\pi^*\flat_{\omega'}\pi_*X$ which vanishes on E. Then $\flat_\omega X - \pi^*\widetilde{\flat_{\omega'}}\pi_*X \in Ann\ TN$ and, if we apply $\sharp_P$ and use the first and the second conditions of (i), we get

$$AX - \sharp_P\pi^*\widetilde{\flat_{\omega'}}\pi_*X \in E \cap TN.$$

Then (3.4) allows us to conclude that $\pi_*AX = A'\pi_*X$, and formula (3.7) turns out to be exactly

$$\iota^*\{\omega,\omega\}_P = \pi^*\{\omega',\omega'\}_{P'}. \tag{3.8}$$

Therefore, $\{\omega,\omega\}_P = 0$ implies $\{\omega',\omega'\}_{P'} = 0$. □

The most interesting case is that of a closed complementary 2-form ω since then $(P, A = \sharp_P \circ \flat_\omega)$ is a Poisson–Nijenhuis structure [9]. In this case, if reduction exists, (Q, P', A') is again a Poisson–Nijenhuis manifold. Following is a particular reduction theorem which refers to this case.

Theorem 3.2. *Let (M, P) be a Poisson manifold, and N a submanifold of M such that $E := \sharp_P Ann\ TN$ has a constant dimension along N, and it satisfies conditions* (i)–(iii) *of the beginning of Section 2, which ensure the existence of a reduction (Q, P') of (M, P) via (N, E). Let ω be a closed complementary 2-form of (M, P) such that E and TN are ω-orthogonal. Then, ω is an $\mathcal{F}$-foliated 2-form, and it projects to a 2-form ω' of Q which is a closed complementary 2-form of (Q, P').*

Proof. First, we notice that the existence of the foliation $\mathcal{F}$, i.e., the fact that $E \cap TN$ is integrable along N is ensured if we ask $dim(E \cap TN) = const.$ ($E = \sharp_P Ann\ TN$). (See, for instance, [7, Proposition 7.17].) Then, for our particular E, the first two conditions of (i) of Theorem 3.1 are satisfied, and the third condition of (i) of Theorem 3.1 is ensured by the ω-orthogonality of E and TN. Hence, by Theorem 3.1, Theorem 3.2 will be proven if we show that ω is $\mathcal{F}$-foliated. Since $\forall X \in T\mathcal{F}$ and $\forall Y \in TN$ we have $\omega(X, Y) = 0$, because of the ω-orthogonality of E and TN, ω is $\mathcal{F}$-foliated iff $X(\omega(Y, Z)) = 0\ \forall X \in T\mathcal{F}$ and for all $\mathcal{F}$-foliated vector fields Y, Z of N. But this latter fact immediately follows from $d\omega(X, Y, Z) = 0$. □

Remark 3.3.

(1) In Theorem 3.2, $A := \sharp_P \circ \flat_\omega$ provides (M, P) with a Poisson–Nijenhuis structure [9]. This structure is reducible in the sense of Theorem 2.1, and the reduced Poisson–Nijenhuis manifold is $(Q, P', A' := \sharp_{P'} \circ \flat_{\omega'})$.

(2) In Theorem 3.2, the foliation $\mathcal{F}$ is given by $T\mathcal{F} = \sharp_P Ann(E + TN)$.

(3) For symplectic manifolds, a closed complementary 2-form is equivalent to a $P\Omega$-structure in the sense of [5] (see [9]). The cases of $P\Omega$-reduction discussed in [5] are contained in Theorems 3.1 and 3.2.

4. Reduction under group actions

Let (M, P) be a Poisson manifold endowed with a Hamiltonian action of a connected Lie group G and an equivariant momentum map $J : M \to \mathcal{G}^*$, where $\mathcal{G}$ is the Lie algebra of G and $\mathcal{G}^*$ is the dual space of $\mathcal{G}$. Let $\gamma \in \mathcal{G}^*$ be a common regular value of the restrictions of J to the symplectic leaves of P such that $M_\gamma := J^{-1}(\gamma) \neq \emptyset$, and it has a clean intersection with the symplectic leaves of P and with the orbits of G in M. Then, it is known (e.g., [7]) that $E = T(Orbits\ G)$ is a vector subbundle of $TM|_{M_\gamma}$ (the orbits of the points of M_γ have all the same dimension equal to the dimension of G), which intersects TM_γ following the tangent bundle of the foliation $\mathcal{F}$ of M_γ by the connected components of the orbits of G_γ := the isotropy subgroup of $\gamma \in \mathcal{G}^*$ for the coadjoint representation of G. Moreover, if $M_\gamma/\mathcal{F}$ is the Hausdorff manifold Q, Q has a Poisson structure P' defined by the reduction of (M, P) via (M_γ, E).

The reduction procedure described above can be extended to a certain type of Poisson–Nijenhuis structures, and this is shown by the following theorem.

Theorem 4.1. *Let $(M, P, G, J, \gamma, Q, P')$ be as described above, and assume that A is a Nijenhuis structure of M which makes (M, P, A) into a Poisson–Nijenhuis manifold, and which is such that:* (i) *at the points of M_γ one has $J_* \circ A = J_*$;* (ii) $\forall \xi \in \mathcal{G}$, $A\tilde{\xi} = \widetilde{C\xi}$, *where tilde denotes the infinitesimal action of G on M, and C an endomorphism of $\mathcal{G}$;* (iii) *A is G-invariant (i.e., $\forall \xi \in G$, $L_{\tilde{\xi}} A = 0$). Then, (P, A) reduces to a Poisson–Nijenhuis structure (P', A') of the reduced Poisson manifold (Q, P').*

Proof. We obtain this result by using Theorem 2.1. The conditions $J_* \circ A = J_*$ and $A\tilde{\xi} = \widetilde{C\xi}$ yield $A(TM_\gamma) \subseteq TM_\gamma$ and $A(E) \subseteq E$ $(E = T(Orbits\, G))$, respectively. It is also known that (2.3) holds in our case [7, p. 112]. Thus, we still have to check that A sends foliated vector fields to foliated vector fields. Since $E = span\{\tilde{\xi}|_{M_\gamma} \mid \xi \in \mathcal{G}\}$, hypotheses (ii) and (iii) easily lead to the fact that (2.9) holds, and the conclusion follows. □

It is also possible to use reduction under a group action in order to reduce complementary 2-forms, and we have:

Theorem 4.2. *Let $(M, P, G, J, \gamma, Q, P')$ be as in Theorem 4.1, and let ω be a complementary 2-form of P on M. Assume that ω is G-invariant, and that the orbits of G are ω-orthogonal to the level sets of the momentum map J. Then, ω is projectable to a 2-form ω' of Q, which is a complementary 2-form of the reduced Poisson structure P'.*

Proof. Now, we use Theorem 3.1 for $N = M_\gamma$ and $E = T(Orbits\, G) = \sharp_P Ann\, TM_\gamma$ [7, formula (7.27)]. Then, since, also, the level set M_γ of J is ω-orthogonal to the orbits of G, all the conditions of (i) of Theorem 3.1 are satisfied. Furthermore, the ω-orthogonality hypothesis yields $i(X)\omega|_{TM_\gamma} = 0\ \forall X \in T\mathcal{F}$. Then, since the leaves of $\mathcal{F}$ are the orbits of the subgroup G_γ of G, and $\forall \xi \in \mathcal{G}$, $L_{\tilde{\xi}}\omega = 0$, it becomes clear that $\iota_N^*\omega$ is $\mathcal{F}$-foliated. Hence, condition (ii) of Theorem 3.1 is also satisfied. □

Furthermore, if we base our argument on Theorem 3.2, instead of Theorem 3.1, we obviously obtain:

Theorem 4.3. *Let $(M, P, G, J, \gamma, Q, P')$ be as in Theorem 4.2, and let ω be a closed complementary 2-form of (M, P) such that the level sets of J and the orbits of G are ω-orthogonal. Then, ω projects to a 2-form ω' of Q which is a closed complementary 2-form of the reduced Poisson structure P'.*

The situation of Theorem 4.3 is interesting since closed complementary 2-forms yield Poisson–Nijenhuis structures. A good example of this situation is obtained as follows. Let $\mathbf{r} \in \wedge^2\mathcal{G}$ be a solution of the Yang–Baxter equation $[\mathbf{r}, \mathbf{r}] = 0$ (see, for instance, [7]). Then, as shown in [9] $\mathbf{r}$ can be interpreted as a closed 2-form on the dual space $\mathcal{G}^*$ which is complementary to the Lie–Poisson structure Π of $\mathcal{G}^*$. Also, since $J : (M, P) \to (\mathcal{G}^*, \Pi)$ is a Poisson map (because J is equivariant), $\omega := J^*\mathbf{r}$ is a closed complementary 2-form of (M, P). Finally, since $i(ker\, J_*)\omega = 0$, the ω-orthogonality hypothesis of Theorem 4.3 is satisfied. Therefore, ω is reducible to Q, and so is the Poisson–Nijenhuis structure $(P, A = \sharp_P \circ \flat_\omega)$.

References

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, 2nd. Ed. (Benjamin/Cummings, Reading, MA, 1978).

[2] I.M. Gel'fand and I.Ya. Dorfman, The Schouten bracket and Hamiltonian operators, Funkt. Anal. Prilozhen. 14(3) (1980) 71–74.

[3] Y. Kosmann-Schwarzbach and F. Magri, Poisson–Nijenhuis structures, Ann. Inst. H. Poincaré, série A 53 (1990) 35–81.

[4] J.L. Koszul, Crochet de Schouten–Nijenhuis et cohomologie, in: *É. Cartan et les mathématiques d'aujourd'hui*, Soc. Math. de France, Astérisque, hors série, 1985, pp.257–271.

[5] F. Magri and C. Morosi, A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson–Nijenhuis manifolds, Quaderno S. Univ. of Milan 19 (1984).

[6] J.E. Marsden and T. Raţiu, Reduction of Poisson manifolds, Lett. Math. Phys. 11 (1986) 161–169.

[7] I. Vaisman, Lectures on the geometry of Poisson manifolds, Progress in Math. Series, Vol. 118 (Birkhäuser, Basel, 1994).

[8] I. Vaisman, The Poisson–Nijenhuis manifolds revisited. Rendiconti Sem. Mat. Torino 52 (1994) 377–394.

[9] I. Vaisman, Complementary 2-forms of Poisson structures. Compositio Math. (1995) to appear.

[10] A. Weinstein, Coisotropic calculus and Poisson groupoids, J. Math. Soc. Japan 40 (1988) 705–727.